On the Reliable Disturbance Decoupling Feedback Equilibrium for Generalized Multi-Channel Systems

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Abstract

This technical note considers the problem of reliable disturbance decoupling for a generalized multi-channel system in a game-theoretic framework. Specifically, we link the problem of stabilization of the multi-channel system to certain properties of controlled invariant subspaces that are associated with the problem of disturbance decoupling, where the structure induced from this family of invariant subspaces is used for a game-theoretic interpretation of the problem. We provide a sufficient condition for the existence of a set of feedback equilibria that maintain the robust stability of the system under possible single-channel controller/agent failure as well as in the presence of unknown disturbances in the system.

Index Terms

Disturbance decoupling, extended LMIs, game theory, multi-channel system, reliable control, stabilization.

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I. INTRODUCTION

In recent years there has been a certain trend towards a reliable control using multi-controller configuration to enhance system robustness/reliability against some changes in operating conditions and/or possible component failures that may occur in actuators/sensors or controllers. In a generalized multi-channel control configuration (e.g., see references [1], [2], [3], [4]), the main objective is to maintain stability and/or certain performance criteria of the closed-loop system both when all of the controllers/agents work together and when some controllers/agents become faulty or deviate from nominal conditions. In this technical note, we consider the problem of reliable disturbance decoupling for a generalized multi-channel system in a game-theoretic framework. Specifically, we link the problem of reliable stability of the multi-channel system to certain properties of controlled invariant subspaces that are associated with the problem of disturbance decoupling, where the structure induced from this family of invariant subspaces is used for the game-theoretic interpretation of the problem, i.e., in the context of noncooperative differential game.

A game-theoretic concept in which the agents follow certain control objectives that are not necessarily associated with standard cost functions can be generalized in various ways. In the context of disturbance decoupling, an interesting extension can be investigated by incorporating the problem of reliable stability to the case where the agents may not able to completely observe the state and/or each other’s actions. Here, the notion of generalized equilibrium can then be introduced to a greater extent by considering games in which the agents know that the other agents will look for a decoupling feedback but are not informed about each other’s decisions. In such a scenario, an equilibrium offers a suitable framework to study an inherent robustness property of the strategies under a family of information structures, since no agent can improve his payoff by deviating unilaterally from this strategy once the equilibrium is attained (e.g., see [5], [6], [7]).

This technical note is organized as follows. In Section II, we present a preliminary result on the problem of reliable stability for a continuous-time linear system using a new class of extended LMIs. Section III presents the main results in a noncooperative game-theoretic framework. A sufficient condition is given in terms of the supremal controlled-invariant subspaces, while a set
of reliable feedback strategies is used to condition the properties of these controlled-invariant subspaces. In Section IV, we present a simple numerical example. Finally, Section V provides some concluding remarks.

Throughout, we use the following notations. For a matrix $A \in \mathbb{R}^{n \times n}$, $\text{He}(A)$ denotes a hermitian matrix defined by $\text{He}(A) = A + A^T$, where $A^T$ is the transpose of $A$. For a matrix $B \in \mathbb{R}^{n \times p}$, $B^\perp \in \mathbb{R}^{(n-r) \times n}$ denotes an orthogonal complement of $B$, which is a matrix that satisfies $B^\perp B = 0$ and $B^\perp B^\perp T > 0$, with $r = \text{rank} B$. $\mathbb{S}_+^n$ denotes the set of strictly positive definite real matrices and $\mathbb{C}^-$ denotes the set of complex numbers with negative real parts, that is $\mathbb{C}^- \overset{\text{def}}{=} \{ s \in \mathbb{C} | \Re\{s\} < 0 \}$. $\sigma(A)$ denotes the spectrum of a matrix $A \in \mathbb{R}^{n \times n}$, i.e., $\sigma(A) \overset{\text{def}}{=} \{ \lambda \in \mathbb{C} | \text{rank}(A - \lambda I) < n \}$ and $\text{GL}_n(\mathbb{R})$ denotes the general linear group consisting of all real nonsingular $n \times n$ matrices. For $\mathcal{B} \subset \mathbb{R}^n$ and $\mathcal{X} \subset \mathbb{R}^n$, the supremal $(A, \mathcal{B})$-controlled invariant subspace contained in $\mathcal{X}$ is denoted by $\sup \mathcal{V}(A, \mathcal{B}; \mathcal{X})$.

II. PRELIMINARY RESULT

Consider the following finite-dimensional generalized multi-channel system

$$\dot{x}(t) = Ax(t) + \sum_{j \in \mathcal{N}} B_j u_j(t) + Ed(t), \quad x(t_0) = x_0, \quad (1)$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$ is the state of the system, $u_j(t) \in \mathcal{U}_j \subset \mathbb{R}^{r_j}$ is the control input to the $j$th-channel of the system, $d(t) \in \mathcal{D} \subset \mathbb{R}^l$ is an unknown disturbance which is not directly observable by all agents, and $t \in [t_0, +\infty)$, and $A \in \mathbb{R}^{n \times n}$, $B_j \in \mathbb{R}^{n \times r_j}$ and $E \in \mathbb{R}^{n \times l}$. $\mathcal{N} \overset{\text{def}}{=} \{1, 2, \ldots, N\}$ represents the set of controllers/agents. Moreover, let the output of the $j$th-agent $y_j(t) \in \mathcal{Y}_j \subset \mathbb{R}^{m_j}$ be given as a linear function of the state, i.e., $y_j(t) = H_j x(t)$ where $H_j \in \mathbb{R}^{m_j \times n}$.

Let $K_j$ be the $j$th-agent’s strategy chosen from a well defined strategy space $\mathcal{K}_j \subset \mathbb{R}^{r_j \times n}$. Let us also introduce some additional notation that will be useful in the sequel

$$r = r_0 \overset{\text{def}}{=} \sum_{j \in \mathcal{N}} r_j, \quad K = K_0 \overset{\text{def}}{=} (K_j)_{j \in \mathcal{N}} \in \mathcal{K}, \quad \mathcal{K} = \mathcal{K}_0 \overset{\text{def}}{=} \prod_{j \in \mathcal{N}} K_j \subset \mathbb{R}^{r \times n},$$

$$r_{-j} \overset{\text{def}}{=} \sum_{i \in \mathcal{N}_{-j}} r_i, \quad K_{-j} \overset{\text{def}}{=} (K_i)_{i \in \mathcal{N}_{-j}} \in \mathcal{K}_{-j}, \quad \mathcal{K}_{-j} \overset{\text{def}}{=} \prod_{i \in \mathcal{N}_{-j}} K_i \subset \mathbb{R}^{r_{-j} \times n},$$

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where the sets $\mathcal{N}_{-j}$ are defined by $\mathcal{N}_{-j} \overset{\text{def}}{=} \mathcal{N} \setminus \{j\}$ for $j = 1, 2, \ldots, N$ and $\mathcal{N}_{-0} \overset{\text{def}}{=} \mathcal{N}$ with cardinality of $|\mathcal{N}_{-0}| = N$ and $|\mathcal{N}_{-j}| = N - 1$, respectively.

For the generalized multi-channel system in (1), we restrict the set $\mathcal{K}$ to be the set of all linear, time-invariant (reliable) stabilizing state-feedback controllers that satisfies

$$\mathcal{K} \subseteq \left\{ (K_1, K_2, \ldots, K_N) \in \prod_{j \in \mathcal{N}} K_j \mid \sigma(A + B_{-j} \circ K_{-j}) \subseteq \mathbb{C}^-, \forall j \in \mathcal{N} \cup \{0\} \right\},$$

(2)

where $B_{-0} = (B_i)_{i \in \mathcal{N}}$, $B_{-j} = (B_i)_{i \in \mathcal{N}_{-j}}$ and $B_{-j} \circ K_{-j} \overset{\text{def}}{=} \sum_{i \in \mathcal{N}_{-j}} B_i K_i$ for $j = 1, 2, \ldots, N$.

\textbf{Remark 1:} In this technical note, we consider the stability of the closed-loop system $(A + B_{-j} \circ K_{-j})$ under nominal operation condition (i.e., when $j = 0$) as well as under a possible single-channel controller/agent failure (i.e., when $j \in \mathcal{N}$).

Let us define the following matrices that will be used in Theorem 1.

For $j = 0$

$$E_{-0} = \begin{bmatrix} I_{n \times n} & I_{n \times n} & \cdots & I_{n \times n} \end{bmatrix}, \quad \langle X_0, X_{-0} \rangle = \text{block diag}\{X_0, X_0, \ldots, X_0\},$$

$$[A, B]_{U_0, L_{-0}} = \begin{bmatrix} A U_0 & B_1 L_1 & B_2 L_2 & \cdots & B_N L_N \end{bmatrix},$$

$$\langle U_0, W_{-0} \rangle = \text{block diag}\{U_0, W_1, W_2, \ldots, W_N\},$$

and for $j \in \mathcal{N}$

$$E_{-j} = \begin{bmatrix} I_{n \times n} & \cdots & I_{n \times n} & I_{n \times n} & \cdots & I_{n \times n} \end{bmatrix},$$

$$\langle X_j, X_{-j} \rangle = \text{block diag}\{X_j, X_j, \ldots, X_j, X_j, \ldots, X_j\},$$

$$[A, B]_{U_j, L_{-j}} = \begin{bmatrix} A U_j & B_1 L_1 & \cdots & B_{j-1} L_{j-1} & B_{j+1} L_{j+1} & \cdots & B_N L_N \end{bmatrix},$$

$$\langle U_j, W_{-j} \rangle = \text{block diag}\{U_j, W_1, \ldots, W_{j-1}, W_{j+1}, \ldots, W_N\}.$$
Next we can characterize the set \( K \) using a new class of extended LMIs as follow.\(^1\)

**Theorem 1:** Suppose the pairs \((A, B_{-j})\) are stabilizable for all \( j \in \mathcal{N} \cup \{0\} \). Then, there exist \( X_j \in S^+_n, \epsilon_j > 0, U_j \in \text{GL}_n(\mathbb{R}), j = 0, 1, \ldots, N \), \( W_i \in \text{GL}_n(\mathbb{R}) \) and \( L_i \in \mathbb{R}^{r_i \times n}, i = 1, 2, \ldots, N \) such that

\[
\begin{bmatrix}
0_{n \times n} & E_{-j} \langle X_j, X_{-j} \rangle \\
\langle X_j, X_{-j} \rangle E_{-j}^T & 0_{(|\mathcal{N}_{-j}|+1)n \times (|\mathcal{N}_{-j}|+1)n}
\end{bmatrix}
+ \text{He}
\begin{bmatrix}
[A, B_{-j}]_{U_j, L_{-j}} \\
-\langle U_j, W_{-j} \rangle
\end{bmatrix}
\begin{bmatrix}
E_{-j}^T & \epsilon_j I_{(|\mathcal{N}_{-j}|+1)n \times (|\mathcal{N}_{-j}|+1)n}
\end{bmatrix}
< 0, \quad (3)
\]

Furthermore, for any family of \( N \)-tuples \((L_1, L_2, \ldots, L_N)\) and \((W_1, W_2, \ldots, W_N)\) as above, and setting \( K_i = L_i W_i^{-1} \) for each \( i = 1, 2, \ldots, N \), the matrices \((A + B_{-j} \circ K_{-j})\) are Hurwitz for all \( j \in \mathcal{N} \cup \{0\} \), i.e., \( \sigma(A + B_{-j} \circ K_{-j}) \subseteq \mathbb{C}^- \), \( \forall j \in \mathcal{N} \cup \{0\} \).

**Proof:** Note that \( \langle X_j, X_{-j} \rangle E_{-j}^T = E_{-j}^T X_j \) and

\[
\begin{bmatrix}
[A, B]_{U_j, L_{-j}} \\
-\langle U_j, W_{-j} \rangle
\end{bmatrix}
= \begin{bmatrix}
I_{n \times n} & [A, B]_{U_j, L_{-j}} \langle U_j, W_{-j} \rangle^{-1}
\end{bmatrix},
\quad \text{def}
\begin{bmatrix}
I_{n \times n} & (A + B_{-j} \circ K_{-j})
\end{bmatrix}, \quad (4)
\]

\[
\begin{bmatrix}
E_{-j} \\
\epsilon_j I_{(|\mathcal{N}_{-j}|+1)n \times (|\mathcal{N}_{-j}|+1)n}
\end{bmatrix}
= \begin{bmatrix}
\epsilon_j I_{(|\mathcal{N}_{-j}|+1)n \times (|\mathcal{N}_{-j}|+1)n} - E_{-j}
\end{bmatrix}, \quad (5)
\]

for \( j = 0, 1, \ldots, N \).

Then, eliminating \( \langle U_j, W_{-j} \rangle \) from (3) by using these matrices, we have the following matrix inequalities

\[
\begin{bmatrix}
I_{n \times n} & [A, B]_{U_j, L_{-j}} \langle U_j, W_{-j} \rangle^{-1}
\end{bmatrix}
\begin{bmatrix}
0_{n \times n} & E_{-j} \langle X_j, X_{-j} \rangle \\
\langle X_j, X_{-j} \rangle E_{-j}^T & 0_{(|\mathcal{N}_{-j}|+1)n \times (|\mathcal{N}_{-j}|+1)n}
\end{bmatrix}
\times
\begin{bmatrix}
I_{n \times n}
\end{bmatrix}
\quad ((U_j, W_{-j})^{-1})^T [A, B]_{U_j, L_{-j}}^T
\quad = \text{He} \left( (A + B_{-j} \circ K_{-j}) X_j \right) < 0, \quad (6)
\]

\(^1\)Recently, a similar extended LMIs condition together with dissipativity-based certifications have been investigated by Befekadu et al. [8] in the context of reliable stabilization for multi-channel systems.
\[
\begin{bmatrix}
\epsilon_j I((|\mathcal{N}_j|+1)n \times (N+1)n) - E_j \\
\langle X_j, X_{-j} \rangle E^T_{-j}
\end{bmatrix}
\begin{bmatrix}
0_{n \times n} \\
\langle X_j, X_{-j} \rangle E^T_{-j}
\end{bmatrix}
\epsilon_j I((|\mathcal{N}_j|+1)n \times (|\mathcal{N}_j|+1)n) - E_j
\]

Hence, we see that equations (6) and (7) exactly state the Lyapunov stability condition with $X_j \in \mathbb{S}_+^n$ and state-feedback gains $K_i = L_i W_i^{-1}$ for $i = 1, 2, \ldots, N$.

Suppose the system in (1) is stable with state-feedback gains $K_i = L_i W_i^{-1}$ for $W_i \in \text{GL}_n(\mathbb{R})$, $i = 1, 2, \ldots, N$. Then, there exist sufficiently small $\epsilon_j > 0$ for $j = 0, 1, \ldots, N$ that satisfy

\[
\text{He} \left( (A + B_{-j} \circ K_{-j})X_j \right) + \frac{1}{2} \epsilon_j \begin{bmatrix} A, B \end{bmatrix}_{X_j, L_{-j}} \langle X_j, X_{-j} \rangle \begin{bmatrix} A, B \end{bmatrix}^T_{X_j, L_{-j}} < 0, \tag{8}
\]

where $[A, B]_{X_j, L_{-j}} = \begin{bmatrix} AX_j & B_1 L_1 & \cdots & B_{j-1} L_{j-1} & B_j L_j+1 & \cdots & B_N L_N \end{bmatrix}$ for $j = 1, 2, \ldots, N$ and $[A, B]_{X_0, L_0} = \begin{bmatrix} AX_0 & B_1 L_1 & B_2 L_2 & \cdots & B_N L_N \end{bmatrix}$.

Note that $\langle X_j, X_{-j} \rangle > 0$ and $\langle X_j, X_{-j} \rangle E^T_{-j} = E^T_{-j} X_j$, employing the Schur complement for (8), then we have

\[
\begin{bmatrix}
\epsilon_j \langle X_j, X_{-j} \rangle \begin{bmatrix} A, B \end{bmatrix}^T_{X_j, L_{-j}} \langle X_j, X_{-j} \rangle \\
\epsilon_j \langle X_j, X_{-j} \rangle \begin{bmatrix} A, B \end{bmatrix}^T_{X_j, L_{-j}} \langle X_j, X_{-j} \rangle
\end{bmatrix} + \text{He} \left( \begin{bmatrix} [A, B]_{X_j, L_{-j}} \langle U_j, W_{-j} \rangle^{-1} \\
-I_{n \times n}
\end{bmatrix} \right)
\times \begin{bmatrix}
\langle X_j, X_{-j} \rangle E^T_{-j} \\
\langle X_j, X_{-j} \rangle E^T_{-j}
\end{bmatrix}
\epsilon_j I((|\mathcal{N}_j|+1)n \times (|\mathcal{N}_j|+1)n) - E_j
\]

Thus, the above expression, i.e., equation (9), implies that (3) holds with $\langle U_j, W_{-j} \rangle = \langle X_j, X_{-j} \rangle$ for $U_j \in \text{GL}_n(\mathbb{R})$ and $j \in \mathcal{N} \cup \{0\}$.

**Remark 2:** We remark that the above extended LMI framework stated in Theorem 1 is useful in the context of reliable control for a system with generalized multi-channel configurations, since the framework effectively separates the design variables from the system data.

Note that the above condition in Theorem 1 guarantees the control/strategy space to be sufficiently *decentralized* for the game-theoretic interpretation meaningful. In the following section, we provide a sufficient condition for the existence of feedback equilibrium when the agents will
look for decoupling feedback strategies that are associated with a family of controlled invariant subspaces.

III. MAIN RESULTS

In this section, we consider the problem of reliable disturbance decoupling for a generalized multi-channel system in a game-theoretic framework. Specifically, each reliable decoupling agent faces with the problem of finding a feedback strategy $K_j \in K_j$, in the presence of other noncooperative agents whose strategies $(K_i)_{i \in \mathcal{N}, -j} \in \mathcal{K}_{-j}$, such that his output is decoupled from the unknown disturbance and when there is a single-channel control/agent failure in the system.

**Problem 1:** Suppose the class of reliable feedback strategies $\mathcal{K}$ is completely characterized by the set of extended LMIs in (3), then find an $N$-tuple $(K_1, K_2, \ldots, K_N) \in \mathcal{K}$ such that the system in (1) is disturbance decoupled for all agents.

The following lemma, which is a well-known result from geometric control theory (e.g., see [9], [10] for a detailed exposition to this theory), will be stated without proof.

**Lemma 1:** Let $A: \mathbb{R}^n \supset \mathcal{X} \to \mathcal{X}$ and $B_{-j}: \mathbb{R}^{r_j} \supset \mathcal{U}_{-j} \to \mathcal{X}$. Then every subspace $\mathcal{K} \in \mathbb{R}^n$ contains a unique supremal $(A, B_{-j})$-invariant subspace that is given by $\mathcal{V}_j^* \overset{\text{def}}{=} \sup \mathcal{I}_j (A, B_{-j}; \mathcal{K})$ for $j = 0, 1, \ldots, N$.

In the following, we consider the case where $\mathcal{K} = \bigcup_{j \in \mathcal{N}} \text{Ker} H_j$ and investigate conditions for which the reliable feedback strategies $K \in \mathcal{K}$ amount to rendering all closed-loop systems $\left\{ A + B_{-j} \circ K_{-j} \right\}_{j=0}^N$ maximally unobservable from $y_j \in \mathcal{Y}_j$ for all $j = 1, 2, \ldots N$. With this, we can then directly link the reliability of the system to certain properties of invariant subspaces that are associated with the problem of disturbance decoupling. Hence, the the structure induced from these lattice invariant subspaces can lead to a game-theoretic interpretation of the problem. Note that if Problem 1 is solvable for each agent, then all agents are able to respond to any strategy of the others in such a way that they can decouple their output from the unknown disturbance, and even in the presence of possible single-channel control/agent failure in the system.
Then, we state the following theorem which is a direct application of Lemma 1.

**Theorem 2:** Let \( \mathcal{V}_j \subset \mathcal{X} \) for \( j = 0, 1, \ldots, N \) and, with minor abuse of notation, write \( \mathcal{B}_{-j} = \text{Im} B_{-j} \). Then, \( \mathcal{V}_j \) is a member of subspace families \( \mathcal{I}_j(A, B_{-j}; \mathcal{X}) \) that preserves the property of \((A, B_{-j})\)-invariance (i.e., \( \mathcal{V}_j \in \mathcal{I}_j(A, B_{-j}; \mathcal{X}) \)), if and only if

\[
A \mathcal{V}_j \subset \mathcal{V}_j + \mathcal{B}_{-j},
\]

for \( j = 0, 1, \ldots, N \).

**Proof:** Suppose \( \mathcal{V}_j \in \mathcal{I}_j(A, B_{-j}; \mathcal{X}) \) and let \( v^j \in \mathcal{V}_j \) for \( j = 0, 1, \ldots, N \), then \( (A, B_{-j} \circ K_{-j})v^j = w^j \) for some \( w^j \in \mathcal{V}_j \), i.e.,

\[
A v^j = w^j - B_{-j} \circ K_{-j} v^j \in \mathcal{V}_j + \mathcal{B}_{-j},
\]

On the other hand, let \( \{v^j_1, v^j_2, \ldots, v^j_{\mu} \} \) be a basis for \( \mathcal{V}_j \) for \( j = 0, 1, \ldots, N \). Suppose (10) holds, then there exist \( w^j_k \in \mathcal{V}_j \) and \( u^j_k \in \mathcal{U}_{-j} \) for \( k \in \{1, 2, \ldots, \mu \} \) such that

\[
A v^j_k = w^j_k - B_{-j} u^j_k, \quad k \in \{1, 2, \ldots, \mu \}.
\]

If we further define the following mapping \( K^0_{-j} : \mathcal{V}_j \to \mathcal{U}_{-j} \)

\[
K^0_{-j} v^j_k = u^j_k, \quad k \in \{1, 2, \ldots, \mu \},
\]

and then by letting \( K_{-j} \) to be any extension of \( K^0_{-j} \) to \( \mathcal{X} \). We, therefore, have \( (A + B_{-j} \circ K_{-j})v^j_k = w^j_k \in \mathcal{V} \), i.e., \( (A + B_{-j} \circ K_{-j})\mathcal{V}_j \subset \mathcal{V}_j \), so that the invariant subspace \( V_j \) satisfies

\[
\mathcal{V}_j \in \mathcal{I}_j(A, B_{-j}; \mathcal{X}), \quad \forall j \in N \cup \{0\}.
\]

We next consider the problem of reliable disturbance decoupling, where \( K_{-j} \in \mathcal{K}_{-j} \) is accessible, but not manipulatable, for the \( j \)th-agent. In such a scenario, we establish the following additional theorem that provides the solvability condition to the problem of reliable disturbance decoupling.

**Theorem 3:** The problem of reliable disturbance decoupling is solvable for each agent if and only if

\[
\bigcap_{j \in N} \mathcal{V}_j^* \equiv \mathcal{V}^* \supset \text{Im} E,
\]

\( \forall j \in N \cup \{0\} \).
where

\[ V^*_j = \sup_j \left( (A + B_{-j} \circ K_{-j}), \text{Im } B_j; \bigcup_{i \in N} \text{Ker } H_i \right), \]  

(14)

for \( j = 1, 2, \ldots, N \) and \( K_{-j} \in \mathcal{K}_{-j} \).

**Proof:** Sufficiency: Note that from Theorem 2, if we choose \( K \in \mathcal{K} \)

\[(A + B_{-j} \circ K_{-j}) V^*_j \subset V^*_j, \]  

(15)

for \( j = 1, 2, \ldots, N \).

Then, using (13), we have

\[ \langle A + B_{-j} \circ K_{-j} \mid \text{Im } E \rangle \subset \langle A + B_{-j} \circ K_{-j} \mid V^*_j \rangle \subset \bigcup_{i \in N} \text{Ker } H_i, \]  

(16)

**Necessity:** If \( K \in \mathcal{K} \) solves the problem of reliable disturbance decoupling. Then, the subspaces

\[ V_j = \langle A + B_{-j} \circ K_{-j} \mid \text{Im } E \rangle, \text{ for } j = 1, 2, \ldots, N, \]  

(17)

belong to \( \sup_j \left( (A + B_{-j} \circ K_{-j}), \text{Im } B_j; \bigcup_{i \in N} \text{Ker } H_i \right) \) and we thus have

\[ \bigcap_{j \in N} V^*_j \overset{\text{def}}{=} V^* \supset \text{Im } E, \]

\[ \blacksquare \]

**Remark 3:** We remark that the above theorem states that the reliable feedback strategies \( \mathcal{K}(V^*) \) with \( V^* = \bigcap_{j \in N} V^*_j \) induce a family of subspaces which preserves the property of \( (A + B_{-j} \circ K_{-j}) \)-invariance for all \( j \in \mathcal{N} \).

Moreover, we have the following lemma that exactly states the link between the family of supremal controlled-invariant subspaces and the reliable state-feedback strategies.

**Lemma 2:** Let \( \{ V_j \}_{j=0}^N \) be a set of supremal invariant subspaces in \( \mathcal{X} \) with respect to the family of systems \( \{ (A, (B_i)_{i \in N_{-j}}) \}_{j=0}^N \). Then, for each \( K_{-j} \in \mathcal{K}_{-j}(V^*_j) \) with \( V^*_j = \bigcap_{i \in N_{-j}} V^*_i \), there exists a feedback strategy \( K_j \in \mathcal{K}_j(V^*_j) \) for the \( j \)-th agent such that the system in (1) is reliably disturbance decoupled for all agents.\(^{2}\)

\[^{2}\text{We remark that the subspace } V^*_0, \text{ which corresponds to the supremal } (A + B_{-0} \circ K_{-0}) \text{-invariance subspace which is contained in } \bigcup_{j \in \mathcal{N}} \text{Ker } H_j.\]

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Proof: Note that from Theorem 1, i.e., from the standard argument of the stabilizability of the pairs \((A, B_{-j})\) for all \(j \in \mathcal{N} \cup \{0\}\), we can always find an \(N\)-tuple \((K_1, K_2, \ldots, K_N) \in \mathcal{K}\) such that (2) holds.

On the other hand, let us define the following family of subspaces

\[
\mathcal{V}_R \defeq \left\{ \bigcap_{j \in \mathcal{N} \cup \{0\}} \mathcal{V}_j \mid \mathcal{V}_j \in \sup_{\mathcal{I}_j} \mathcal{J}_j(A, B_{-j}; \mathcal{X}), \quad \forall j \in \mathcal{N} \cup \{0\} \text{ and } \exists K \in \mathcal{K} \right\}, \tag{18}
\]

Suppose the subspace \(\mathcal{V}^*\) exists, then it is the largest unique member of the family defined in (18), i.e.,

\[
\mathcal{V}^* = \bigcap_{j \in \mathcal{N} \cup \{0\}} \mathcal{V}^*_j \subseteq \mathcal{V}_R, 
\]

with \(\mathcal{V}^*_j = \sup_{\mathcal{I}_j} \mathcal{J}_j ((A + B_{-j} \circ K_{-j}), \text{Im } B_j \bigcup_{i \in \mathcal{N}} \text{Ker } H_i)\) for all \(j \in \mathcal{N} \cup \{0\}\).

Note that we have \(\mathcal{V}_j = \langle A + B_{-j} \circ K_{-j} | \text{Im } E \rangle\) which further implies that \(\text{Im } E \subseteq \mathcal{V}^*\). Moreover, the induced maps in \(\mathcal{X}/\mathcal{V}^*_j\) and \(\mathcal{X}/\mathcal{V}^*_j\) admit an enveloping lattice for \(\mathcal{V}^*_j\) for all \(j \in \mathcal{N} \cup \{0\}\).3

Hence, we immediately see that the \(N\)-tuple \((K_1(\mathcal{V}^*_1), K_2(\mathcal{V}^*_2), \ldots, K_N(\mathcal{V}^*_N)) \in \mathcal{K}(\mathcal{V}^*)\) satisfies the feedback equilibrium, i.e., a reliable disturbance decoupling for the generalized multi-channel system.

Note that the above lemma implicitly requires that each agent to choose a strategy \(K_j \in \mathcal{K}_j\) for all \(j \in \mathcal{N}\) in such a way no individual agent has an incentive to change his own strategy from the feedback equilibrium. Moreover, the feedback equilibrium is well defined only when each agent can estimate his opponents’ strategies and evaluates his own strategy exactly.

However, a more realistic game model must include the possibility that any information may contain uncertainty such as observation or estimation errors. In the following, we focus on a

3We remark that the set \(\mathcal{Y} \defeq \{\mathcal{V}_j\}_{j=0}^N \in \mathcal{X}\) is a lattice of invariant subspaces since \(\mathcal{V}_j + \mathcal{V}_{-j} \in \mathcal{Y}\) and \(\mathcal{V}_j \cap \mathcal{V}_{-j} \in \mathcal{Y}\) for all \(j \in \mathcal{N} \cup \{0\}\). Note that we can define the gap \(g_j(\mathcal{V}_0, \mathcal{V}_j)\) between the invariant subspaces \(\mathcal{V}_0\) and \(\mathcal{V}_j\) as

\[
g_j(\mathcal{V}_0, \mathcal{V}_j) = \|P_{\mathcal{V}_0} - P_{\mathcal{V}_j}\|, \quad \text{for } j = 1, 2, \ldots, N,
\]

where \(P_{\mathcal{V}_0}\) and \(P_{\mathcal{V}_j}\) are orthogonal projectors on \(\mathcal{V}_0\) and \(\mathcal{V}_j\), respectively. Moreover, it is well known that the set of all controlled invariant subspaces in \(\mathcal{X}\) is compact metric spaces with the gap, which is defined above, as the metric (e.g., see [11], [12]).
game with uncertainty and/or incomplete information. In this context, we introduce the following assumptions:

- A parameter set \( Q \subset \mathbb{R}^\nu \) in which the family of systems \( (A(q), (B_j(q))_{j\in\mathcal{N}}, (H_j(q))_{j\in\mathcal{N}}) \) depends on an uncertain parameter \( q \in Q \), and
- A strategy space \( \hat{\mathcal{K}}_{-j} \subset \mathcal{K}_{-j} \) with \( \hat{\mathcal{K}}_{-j} = \mathcal{K}_{-j} + \delta \mathcal{K}_{-j} \in \hat{\mathcal{K}}_{-j} \) where an uncertainty term \( \delta \mathcal{K}_{-j} \in \Delta \mathcal{K}_{-j} \subseteq \mathbb{R}^{r_j \times n} \) which is associated with \( j \)th-agent’s observation about the others’ strategies.

We further assume the following statements about each agent.

(A1) The \( j \)th-agent may not know exactly the other agents’ strategies \( \mathcal{K}_{-j} \); however, he can estimate their strategies from a non-empty and compact strategy space \( \hat{\mathcal{K}}_{-j} \subset \mathcal{K}_{-j} \).

(A2) The supremal invariant subspace \( \mathcal{V}_j^\star \) which involves an uncertain parameter \( q \in Q \) and uncertain terms \( \delta \mathcal{K}_{-j} \in \Delta \mathcal{K}_{-j} \) is given by

\[
\mathcal{V}_j^\star(q, \delta \mathcal{K}_{-j}) \in \sup \mathfrak{I}_j \left( (A(q) + B_{-j}(q) \circ (\mathcal{K}_{-j} + \delta \mathcal{K}_{-j})), \operatorname{Im} B_j(q) ; \bigcup_{i \in \mathcal{N}} \ker H(q)_i \right),
\]

for \( j = 1, 2, \ldots, N \). Here, we also assume that the \( j \)th-agent does not know the exact values of \( q \) and \( \delta \mathcal{K}_{-j} \), but he can estimate both from \( Q \) and \( \Delta \mathcal{K}_{-j} \).

(A3) The family of systems \( (A(q), (B_j(q))_{j\in\mathcal{N}}, (H_j(q))_{j\in\mathcal{N}}) \) belongs to some non-empty and compact set for all \( q \in Q \).

To solve this family of problems involving terms \( q \) and \( \delta \mathcal{K}_{-j} \) (i.e., based on Assumption (A1)–(A3)), we require each agent to consider the worst-case \( \mathcal{V}_j^\star \) which is defined by

\[
\mathcal{V}_j^\star(q, \delta \mathcal{K}_{-j}) \equiv \inf \left\{ \mathcal{V}_j^\star(q, \delta \mathcal{K}_{-j}) \mid q \in Q \text{ and } \delta \mathcal{K}_{-j} \in \Delta \mathcal{K}_{-j} \right\}.
\]

Then, we can define the robust feedback equilibrium for the generalized multi-channel system as follows.\(^4\)

\(^4\)Here we remark that such problem formulation is considered as a complete information game under the family of invariant subspaces \( \{ \mathcal{V}_j \}_{j=1}^N \).
Definition 1: Suppose $\tilde{V}_j^*$ is defined by (20). Then the $N$-tuple $(K_j)_{j \in \mathcal{N}} \in \mathcal{K}$ is called a robust disturbance decoupling feedback equilibrium for the multi-channel system, if there exists a family of robust supremal $(A(q), B_{-j}(q))$-invariant subspaces relative to $\mathcal{Q}$ and $\Delta \mathcal{K}_{-j}$ for all $j \in \mathcal{N}$.

We next provide a sufficient condition for the existence of a robust feedback equilibrium for the generalized multi-channel system. Note that the strategy space $\hat{\mathcal{K}}_{-j} \subseteq \mathcal{K}_{-j}$ (see Assumption (A2)) can be considered as a set-valued mapping $\hat{\mathcal{K}}_{-j} : \mathcal{K}_{-j} \subseteq \mathcal{K}_{-j} \to \mathcal{K}_{-j}$ (e.g., see [13]).

Assumptions (A1) - (A3) above imply that the supremal invariant subspace $\tilde{V}_j^*$ has the following properties.

(P1) $\tilde{V}_j^*$ is finite for any $(K_j, \mathcal{K}_{-j}) \in \mathcal{K}$ for $j = 1, 2, \ldots, N$.

(P2) For any fixed $\mathcal{K}_{-j} \in \mathcal{K}_{-j}$, the set $\tilde{V} \overset{\text{def}}{=} \{\tilde{V}_j^*\}_{j=0}^N \subset \mathcal{X}$ is a lattice of invariant subspaces.

Based on these, we can obtain the following lemma for the existence of a robust feedback equilibrium.

**Lemma 3:** Suppose the following statements hold

(i) The strategy space $\mathcal{K}_j$ is non-empty and compact for every agent $j \in \mathcal{N}$, and

(ii) The set $\tilde{V} = \bigcap_{j \in \mathcal{N} \cup \{0\}} \tilde{V}_j^*$ includes the subspace $\text{Im } E$.

Then, there exists at least one feedback equilibrium that maintains the robust stability of the system under possible single-channel controller/agent failure as well as in the presence of unknown disturbances in the system.

**Proof:** From Assumptions (A1) - (A3), for any $j \in \mathcal{N}$, the worst-case supremal invariant subspace $\tilde{V}_j^*$ is finite for any $\mathcal{K}_{-j} \in \mathcal{K}_{-j}$. Note that the set $\tilde{V} = \{\tilde{V}_j^*\}_{j=0}^N$ is a lattice of invariant subspaces for $\mathcal{K} \in \mathcal{K}$, and it is also assumed to contain $\text{Im } E$, i.e., $\tilde{V} \supset \text{Im } E$. As a consequence, there exists at least one reliable disturbance decoupling feedback equilibrium that preserves the stability of the family of systems $(A(q), (B_{j}(q)))_{j \in \mathcal{N}}, (H_{j}(q))_{j \in \mathcal{N}}$ under possible single-channel controller/agent failure as well as in the presence of unknown disturbances in the system.
We remark that for a generalized uncertain multi-channel system (i.e., a family of systems \( \{ A(q), B_j(q) \}_{j \in \mathcal{N}}, (H_j(q))_{j \in \mathcal{N}} \) that depends only on the uncertain parameter \( q \in \mathbb{Q} \subset \mathbb{R}^\nu \), while each agent exactly knows the other agents strategies, the family of robust supernal \((A(q), B_{-j}(q))\)-invariant subspaces for all \( j \in \mathcal{N} \cup \{0\} \) relative to \( \mathbb{Q} \) can then be treated in a similar way as that presented in [14]. We also remark that such a noncooperative disturbance decoupling problem in the context of differential game has been considered in the paper [15] using techniques from geometric control theory, but in different context.

IV. Numerical example

Consider the following simple example where the system matrices are given by

\[
A(q) = \begin{bmatrix}
0 & 1 + q_1 + q_2 & 0 \\
-1 - q_1 & q_2 & 1 + q_2 \\
0 & -1 - q_1 - q_2 & 0
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}, \quad B_3 = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \quad E = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}.
\]

We further assume that the system matrix \( A(q) \) depends on the uncertain parameter \( q \in \mathbb{Q} \) where \( \mathbb{Q} \defeq \{ q \in \mathbb{R}^2 \mid q_i \in [\underline{q}_i, \overline{q}_i], i = 1, 2 \} \), whereas the other matrices are assumed to be known exactly by all agents.

Let the agents’ output matrices be given as follows

\[
H_1 = \begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad H_2 = \begin{bmatrix}
-1 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}, \quad H_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 1
\end{bmatrix}.
\]

and let the uncertain parameter \( q = (p_1, p_2) \) be in a compact set \( \mathbb{Q} \defeq [-0.125, 0.125] \times [-0.25, 0.25] \).

Furthermore, the subspaces \( \bigcup_{j \in \mathcal{N}} \text{Ker} \ H_j, \mathcal{N} \defeq \{1, 2, 3\} \), and \( \text{Im} \ E \) are given as follows

\[
\bigcup_{j \in \mathcal{N}} \text{Ker} \ H_j = \begin{bmatrix}
0.7071 & 0.5774 & 0.4082 \\
-0.7071 & 0.5774 & 0.4082 \\
0.0000 & -0.5774 & 0.8165
\end{bmatrix}, \quad \text{Im} \ E = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}.
\]
In what follows, we introduce a family of subspaces (c.f. Equation (18)) whose structure is associated with the problem problem of robust disturbance decoupling feedback equilibrium for the above system.

\[ \tilde{V}_R \overset{\text{def}}{=} \bigcap_{j \in \mathcal{N} \cup \{0\}} \tilde{V}_j \subset \mathcal{X} \mid \exists K \in \mathcal{K}, \text{ s.t. (} A(q) + B_{-j} \circ K_{-j} \text{)} \tilde{V}_j \subset \tilde{V}_j, \]

\[ \forall q \in \mathcal{Q} \text{ and } \forall j \in \mathcal{N} \cup \{0\} \].

Note that each reliable decoupling agent is facing with the problem of finding a feedback strategy \( K_j \in \mathcal{K}_j \) for \( j \in \mathcal{N} \) that

(i) preserves the stability of the system in the presence of other noncooperative agents whose strategies \((K_i)_{i \in \mathcal{N} \setminus j} \in \mathcal{K}_{-j}\) and/or model uncertainty and,

(ii) decouples his output from the unknown disturbance \( d(t) \in \mathbb{R} \) for \( t \in [0, \infty) \) even when there is a single-channel control/agent failure in the system.

Then, using Theorem 3, we compute the invariant subspaces in (19) (or (20)) noting that the condition of (13) must hold

\[ \tilde{V}_* = \bigcap_{j \in \mathcal{N} \cup \{0\}} V_j^* = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}, \]

which indeed contains the subspace \( \text{Im } E \).

For \( \epsilon_j = 1, j \in \mathcal{N} \cup \{0\} \) (see Theorem 1)\(^5\), the stabilizing feedback equilibrium solutions that achieve robust/reliable disturbance decoupling are \( K_1 = [\begin{array}{ccc} -1.4647 & 0.1036 & 0.0249 \\ -0.1285 & -1.62219 & 0.1051 \end{array} \], K_2 = [\begin{array}{ccc} 0.0301 & -0.1694 & -1.5052 \end{array} \], \) and \( K_3 = [\begin{array}{ccc} 0.0301 & -0.1694 & -1.5052 \end{array} \] ); and, moreover, the corresponding gap distances \( \rho_j(\mathcal{Y}_0, \mathcal{Y}_j) \) between the invariant subspaces \( \mathcal{Y}_0 \) and \( \mathcal{Y}_j \) for \( j = 1, 2, 3 \) are zero; and this exactly shows the minimum distance between the lattice of invariant subspaces \( \{\mathcal{Y}_j\}_{j \in \mathcal{N} \cup \{0\}} \), which was mentioned in the context of game-theoretic framework in the previous section (cf. footnote 3 of Section III).

\(^5\)Note that, for the family of systems \( (A(q), (B(q)_j)_{j \in \mathcal{N}}, (H(q)_j)_{j \in \mathcal{N}}), q \in \mathcal{Q} \), we cannot design reliable disturbance decoupling state-feedback controllers based on a common solution of Lyapunov inequalities, i.e., we cannot find a solution set \( \{X, K_1, K_2, K_3\} \), with \( X \in \mathbb{S}_{++}^3 \) and \( K \overset{\text{def}}{=} (K_1, K_2, K_3) \in \mathcal{K} \), which satisfies the conditions in (6) and (7).
V. Concluding Remarks

In this technical note, we considered a reliable disturbance decoupling problem for a generalized multi-channel system in a game-theoretic framework. A sufficient condition for the solvability of the problem are derived in terms of a family of supremal controlled-invariant subspaces, while a set of reliable/robust feedback strategies is used to condition the properties of these controlled-invariant subspaces; consequently, we verified that the problem of reliable disturbance decoupling has a robust feedback equilibrium both when the multi-channel system belongs to some compact uncertainty set and when its family of controlled invariant subspaces forms a complete lattice.

References


