Input-to-State Stability of Hybrid Systems with Receding Horizon Control in the Presence of Packet Dropouts

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Abstract

We analyze the stability of a constrained piecewise continuous hybrid system that is controlled by a receding horizon controller across an unreliable communication channel. We assume the presence of a buffer at the actuator to store the control input sequence received from the controller to compensate for any possible packet dropouts in future transmissions. Input-to-state stability is considered under the assumption that the number of consecutive packet dropouts is bounded using tightened state constraints in the optimization problem solved by the controller.

1 Introduction

Systems in which the plant and controller are connected across a communication channel are now a standard research direction. We analyze the input-to-state stability of a discrete-time constrained piecewise continuous hybrid system that is controlled by a receding horizon controller across a communication channel that can erase data packets. Receding horizon control, or model predictive control, (see e.g., Bertsekas (2005); Mayne et al (2000); Scokaert et al (1999)) is a control algorithm that relies on solving finite horizon optimal control problems successively. The advantage of receding horizon control is that state and input constraints can be easily considered. Constrained piecewise continuous system models provide a powerful tool to model hybrid systems (see, e.g. Heemels et al (2001), see also works such as Antunes et al (2010) for control of stochastic hybrid systems in a networked setting). However, since the dynamics is not a continuous function, the stability analysis for such systems can get quite involved. Our chief technical tool is the use of tightened state constraints (see, e.g. Lazar et al (2006, 2007); Lazar and Heemels (2009); Limon et al (2002)) in receding horizon control. Our work is most closely related to Pin and Parisini (2011) and Quevedo and Nesic (2011). The chief difference is that we consider piecewise continuous systems. Since the overall system is not continuous, the tightened state constraints need to be modified to ensure stability. Using appropriately defined tightened constraints, robust feasibility and stability in the presence of both discontinuities and packet dropouts can be guaranteed.

The paper is organized as follows. In Section 2, the problem formulation is presented. The stability analysis is provided in Section 3. Finally, we conclude in Section 4.

Notation: The Euclidean norm is denoted by $|| \cdot ||$. Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{> 0}$ denote the set of real numbers, non-negative real numbers, and positive real numbers, respectively, and $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, and $\mathbb{Z}_{\geq 1}$ denote the set of integer, non-negative integers, and positive integers, respectively. For a sequence $\{s_k\}_{k\in \mathbb{Z}_{\geq 0}}$, let $s_k^+$ denote the set $\{s_k, s_{k+1}, \ldots, s_j\}$ and $s_k^+$ be an empty set. We denote the identity function by $\text{Id}()$. A function $\varphi: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $K$ function if it is continuous, strictly increasing, and $\varphi(0) = 0$. It is of class $K_0\infty$ if it is of class $K$ and unbounded. A function $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{> 0} \to \mathbb{R}_{\geq 0}$ is a $KL$ function if for any fixed $k$, $\beta(\cdot, k)$ is a $K$ function and for every $j \in \mathbb{R}_{\geq 0}$, $\lim_{k \to 0} \beta(j, k) = 0$. For two arbitrary sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$, $A \sim B := \{c \in \mathbb{R}^n|c + B \subseteq A\}$. For a set $\mathbf{W}$, $\mathbf{W}^i$ is the $i$-th cartesian power of $\mathbf{W}$.

2 Problem Formulation

Consider a discrete-time piecewise continuous system

$$x(k + 1) = f_j(x(k), u(k), w(k)), \quad \forall x(k) \in \Omega_j,$$  

where $f_j: \Omega_j \times U \times \mathbf{W} \to \mathbb{R}^n$ is a continuous, possibly nonlinear function in $x$. The states and inputs are
restricted to sets $X \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^p$ (referred to as state constraints and control input constraints respectively) with $0 \in X$ and $0 \in U$. The index $j$ belongs to a finite index set $\Xi := \{1, \ldots, \xi\}$ such that the collection $\{\Omega_j \subseteq \mathbb{R}^n | j \in \Xi\}$ partitions $X$. We assume that there exists $\Omega_{j_0}$ such that $0$ belongs to the interior points of $\Omega_{j_0}$, and $f_{j_i}(0, 0, 0) = 0$. The disturbance set $W$ is a subset of $\mathbb{R}^n$ with each of its elements having Euclidean norm less than a positive constant $\mu$.

**Assumption 1** For each $j \in \Xi$, there exists a $K$-function $\eta$ such that for all $x, y \in \Omega_j$, $u \in U$, $w \in W$, $t \geq 0$ and $i \geq 0$,

$$f_j(x, u, w) - f_j(y, u, 0) \leq \eta(||x - y||) + \mu,$$  

(2)

Notice that one can, e.g., consider $\eta(\cdot) := \max_{j \in \Xi} \eta_j(\cdot)$ where we have $\|f_j(x, u, w) - f_j(y, u, 0)\| \leq \eta_{j_0}(\|x - y\|) + \mu$, for all $x, y \in \Omega_j$, $u \in U$, $w \in W$, $j \in \Xi$. For future reference, define $\tilde{\eta}(0) := \eta \circ \tilde{I}(\cdot)$ recursively, for $i = 1, \ldots, N - 1$, with $\tilde{\eta}(0) := I(d)(\cdot)$.

**Control Packet Design:** We use receding horizon control to compute the control sequences. Let $x_{i,k}$ (resp. $u_{i,k}$) denote the predicted state (resp. control input) at time $k$ given the initial condition $x(k)$ at time instant $k$. Note that $x_{0,k} := x(k)$. At each time $k$, the controller solves the minimization problem:

$$\min_{\{u_{i,k}\}_{i=0}^{N-1}} \sum_{i=0}^{N-1} L(x_{i,k}, u_{i,k}) + F(x_{N,k})$$

subject to $x_{i+1,k} = f_j(x_{i,k}, u_{i,k}, 0), x_{i,k} \in \Omega_j$

$$x_{i,k} \in X^i, \quad i = 1, \ldots, N - 1$$

$$u_{i,k} \in U^i, \quad i = 0, 1, \ldots, N - 1$$

$$x_{N,k} \in X_T,$$

where $N$ is the prediction horizon, $L(\cdot, \cdot)$ is a non-negative function with $L(0, 0) = 0$ denoting the stage cost, $F(\cdot)$ is a non-negative function with $F(0) = 0$ that denotes the final state cost function, and $X_T$ is the terminal state constraint. The details about how to choose the penalty functions and the final state constraint are given in Section 3. In order to compensate for the packet dropout effect and discontinuity of the dynamics, we adopt the tightened constrained set $X^i := \bigcup_{j \in \Xi} \{\Omega_j \sim L^i\}$ where the margin set $L^i := \{\zeta \in \mathbb{R}^n | ||\zeta|| \leq \sum_{j=1}^{\xi} \tilde{\eta}(0)(\mu)\}$. Note that this definition is more conservative than the one used in Lazar and Heemels (2009) so as to compensate for the imperfect communication channel. Let $V^*(x(k))$ be the optimal cost function value obtained by solving the problem (3). We adopt the following assumptions commonly made in model predictive control (see, e.g. Lazar et al. (2006)-Limon et al. (2002)):

**Assumption 2** There exist $K$ functions $\alpha_F(s) := \tau s^\lambda$, $\alpha_1(s) := a s^\lambda$, and $\alpha_2(s) := b s^\lambda$, where $\tau$, $a$, $b$, and $\lambda \in \mathbb{R}_{>0}$, such that

(1) $L(x, u) \geq \alpha_1(||x||), \forall x \in X, u \in U$,

(2) $|F(x) - F(y)| \leq \alpha_F(||x - y||), \forall x, y \in X_F(N)$,

(3) $V^*(x) \leq \alpha_2(||x||), \forall x \in X_F(N)$, where $X_F(N)$ is the feasible set of the optimization problem (3) with $0 \in X_F(N)$.

**Network and Buffering Model:** At every time $k$, the controller transmits the sequence of control inputs $U(k) := \{u_{0,k}, u_{1,k}, \ldots, u_{N-1,k}\}$ to the plant across an analog erasure channel. The effect of the channel can be described by a dropout sequence $\{d(k)\}_{k \in \mathbb{Z}_{\geq 0}}$, defined as

$$d(k) = \begin{cases} 1 & \text{if packet dropout occurs at time } k \\ 0 & \text{if no packet dropout at time } k \end{cases}$$

(4)

The next assumption follows Quevedo and Nesic (2011).

**Assumption 3** The number of consecutive packet dropouts is bounded by $N - 1$, i.e., $m_i \leq N - 1, \forall i \in \mathbb{Z}_{\geq 0}$.

If $d = 0$, the actuator receives the sequence $U(k)$. We assume the presence of a buffer of length $N$ at the actuator. The actuator stores the sequence $U(k)$ in the buffer, overwriting any existing contents. On the other hand, if $d = 1$, the buffer is shifted by one. More specifically, the buffer sequence $b(k)$ is defined as follows:

$$b(k + 1) = (1 - d(k))U(k) + d(k)Sb(k - 1),$$

(5)

$$S := \begin{bmatrix} 0_p & I_p & 0 & \ldots & 0_p \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_p & \ldots & 0_p & I_p & 0_p \\ 0_p & \ldots & 0_p & I_p & \ldots & 0_p \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0_p & \ldots & 0_p & \ldots & 0_p \end{bmatrix}$$

(6)

At each time $k$, the first element of the buffer sequence $b(k)$ is the control input $u(k)$ applied to the plant. Denote the $i$-th successful transmission time instant by $k_i$ and the number of consecutive packet dropouts as $m_i := k_{i+1} - k_i - 1, i \in \mathbb{Z}_{\geq 0}$.

**Problem Specification:** Given this setup, we wish to analyze the stability of the receding horizon control (3) applied to the piecewise continuous system (1). Specifically, we use the input-to-state stability notion.

3 Stability Analysis

Our chief technical tool is the use of tightened state constraints to compensate for the disturbances due to the piecewise continuous process model and the control input sequence dropouts. Following Lazar and Heemels (2009), we make the following assumption.
Assumption 4 There exist $N \in \mathbb{Z}_{\geq 1}$, $\theta > \theta_1 > 0$, $\mu > 0$, a $K_\infty$ function $\psi_V$, and a locally stabilizing control law $h(\cdot) : \mathbb{R}^n \to \mathbb{R}^p$ that ensures the stability of the system, such that

(1) $\phi(\eta^N(\mu)) < \theta - \theta_1$,
(2) $F_\theta \subset \{ x \in \mathbb{R}^n | F(x) \leq \theta \} \subset \Omega_{\theta_1} \sim L^N_\mu \cap X_U$, where $X_U := \{ x \in X|h(x) \in U \}$,
(3) $f_\theta(x, h(x)) \in F_\theta$, $\forall x \in F_\theta$,
(4) $f(F_j(x, h(x))) - f(F_i(x) + L(x, h(x))) \leq 0$, $\forall x \in F_\theta$,
(5) $|V^*(x) - V^*(z)| \leq \psi_V(||x - z||)$, $\forall x, z \in X_f(N)$,

Note that Assumption 4.5 is not included in Lazar and Heemels (2009). We require it since our problem formulation involves the possibility of packet dropouts. To proceed, we define the following two mappings:

- Denote by $f^i(\cdot, \cdot)$ the state evolution from time $k$ with the predicted control input $U_k$:
  
  $f^i(x(k), w_0^{i-1}) = f_j(f^i(x(k), w_{i-1}^{i-2}), u_{i-1,k}, w(i-1)),$
  $i = 1, 2, ..., N$,

- $f_0(x(k), w_0^{i-1}) := x(k)$. The index $i$ indicates the $i$-th iterated mapping.

Denote by $f^i(\cdot)$ the state evolution from time $k$ with the predicted control input $U_k$, but without any process noise present. Thus, $f^i(x(k)) := f^i(x(k), 0)$ and $f^0(x(k)) := x(k)$. The difference between these two mappings is bounded.

Lemma 5 Given any $x \in X_f(N)$, if Assumption 1 is satisfied, and the nominal trajectories follow the index sequences $\{j_1, ..., j_N\}$, i.e. $f^i(x) \in \Omega_{j_i} \sim L^i_\mu$, for $i = 1, ..., N$, then

\[
(f^i(x), f^i(x, w_0^{i-1})) \in \Omega_{j_i} \times \Omega_{j_i}, i = 1, ..., N - 1,
\]

\[
||f^i(x) - f^i(x, w_0^{i-1})|| \leq \eta^i(\mu), i = 1, ..., N.
\]

PROOF. The proof is through induction. For $i = 1$, since $f^1(x, w(0)) = f^1(x) + w(0)$, and $f^1(x) \in \Omega_{j_1} \sim L^1_\mu$, (9) and (10) obviously hold. Now assume (9) and (10) hold for $1 \leq i - 1 \leq N - 2$. Then, $||f^i(x) - f^i(x, w_0^{i-1})|| \leq \eta^i(\mu) + \mu = \bar{\eta}^i(\mu) \leq \sum_{p=1}^{i} \bar{\eta}^p(\mu)$. This implies that $f^i(x, w_0^{i-1}) - f^i(x) \in L^i_\mu$, which proves (10). Furthermore, since $f^i(x) \in \Omega_{j_i} \sim L^i_\mu$, we conclude that $f^i(x, w_0^{i-1}) \in \Omega_{j_i}$. Thus, by induction, we can conclude that (9) and (10) are true for $1 \leq i \leq N - 1$. For $i = N$, since $(f^{N-1}(x), f^{N-1}(x, w_0^{N-2})) \in \Omega_{j_{N-1}} \times \Omega_{j_{N-1}}$, then $||f^N(x) - f^N(x, w_0^{N-1})|| \leq \eta^N(\mu) + \mu = \bar{\eta}^N(\mu) \leq \sum_{p=1}^{N} \bar{\eta}^p(\mu)$. Thus (10) holds for $i = N$.

The next three results prove a robust positive invariance property of the feasible set $X_f(N)$. Note that while in the absence of packet dropouts, robust positive invariance is required only for the set $X_f(N)$ for $f^1(x, w(0))$, we require the property to hold for up to $N$ steps (see also Quevedo and Nesić (2011)). Let $(f^i(x(k)), ..., f^N(x(k)))$ be a state evolution of the nominal system, obtained by applying the predicted control input $U(k) = (u_0, u_{1,k}, ..., u_{N-1,k})$ and from the initial state $x(k)$. Let $j_1, ..., j_N$ be the corresponding index sequences, i.e. $f_i(x(k)) \in \Omega_{j_i} \sim L^i_\mu$, for $i = 1, ..., N$. Let $(\bar{\phi}^1, \bar{\phi}^2, ..., \bar{\phi}^N)$ be also a state evolution of the nominal system, obtained by applying the shifted control input sequence $(u_0, u_{1,k}, ..., u_{N-1,k}, h(\phi^{N-d}), ..., h(\phi^{N-d-1}))$ and from the initial condition $\bar{\phi}^0 := f^0(x(k), w_0^{d-1})$, where $d \leq N$ is a positive constant and $h$ is the stabilizing control law introduced in Assumption 4.

Lemma 6 Given any $x \in X_f(N)$, suppose Assumption 1 is satisfied, and the nominal trajectories follow the index sequences $(j_1, ..., j_N)$. Then $f^{i+d}(x(k)) \in \Omega_{j_{i+d}} \sim L^d_\mu$, it implies that $\bar{\phi}^i \in \Omega_{j_{i+d}}$. Thus, the properties hold for $i = 0$. Now assume the properties hold for $0 \leq i - 1 \leq N - (d + 2)$. Then

\[
||\bar{\phi}^i - f^{i+d}(x(k))|| = ||f_{j_{i+d-1}}(\bar{\phi}^{i-1}, u_{i+d-1,k}, 0)||
\]

\[
- f_{j_{i+d-1}}(f^{i+d}(x(k)), u_{i+d-1,k}, 0)||
\]

\[
\leq \eta \circ \bar{\eta}^{i+d-1}(\mu) \leq \bar{\eta}^{i+d}(\mu) \leq \sum_{p=1}^{i+d} \bar{\eta}^p(\mu),
\]

where the first inequality follows from Assumption 1. It follows that $\bar{\phi}^i - f^{i+d}(x(k)) \in L^d_\mu$. Since $f^{i+d}(x(k)) \in \Omega_{j_{i+d}} \sim L^d_\mu$, we can conclude that $\bar{\phi}^i \in \Omega_{j_{i+d}}$. By the principle of induction, we have thus proven the properties hold for $0 \leq i \leq N - (d + 1)$. For $i = N - d$, since $f^{N-1}(x(k)) \in \Omega_{j_{N-1}} \sim L^{N-d+1}_\mu$, then $||f^N(x(k)) - \bar{\phi}^{N-d}|| \leq \bar{\eta}^N(\mu)$.

Lemma 7 If $x \in \Omega_i \sim L^{i+d}$, and $||y - x|| \leq \bar{\eta}^{i+d}(\mu)$, then $y \in \Omega_i \sim L^i_\mu$.

PROOF. The proof is similar to that of (Lazar and Heemels, 2009, Lemma 10).

Proposition 8 If Assumptions 1-4 are satisfied, and $X_T = F_\theta_1$, then starting from any initial condition $x(k) = x \in X_f(N)$, the open-loop mapping $f^i(x, w_0^{i-1}) \in X_f(N)$, $\forall w_0^{i-1} \in W^i$, $\forall i = 1, 2, ..., N$.
PROOF. Given any \( x \in X_f(N), \tilde{f}^{i+d}(x) \in \Omega_{i+d} \sim L_{i+d} \). Thus, for \( i = 1, \ldots, N - (d + 1), \) Lemmas 6 and 7 yield that \( \bar{\phi}_i \in \Omega_{i+d} \sim L_{i+d} \in X^1 \), i.e. the state constraints are satisfied for \( i = 1, 2, \ldots, N - (d + 1) \). For \( i = N - d \), Lemma 6 yields \( \| \tilde{\phi}^{N-d} - \bar{f}^N(x) \| \leq \tilde{\gamma}^N(\mu) \). It follows from Assumption 2.2 that \( F(\tilde{\phi}^{N-d}) - F(\bar{f}^N(x)) \leq \alpha\tilde{\gamma}^{N}(\mu) \). Since from Assumption 4.1, \( \alpha\tilde{\gamma}^{N}(\mu) \leq \theta - \theta_1 \), and \( X_T = F_\theta \), this implies that \( F(\tilde{\phi}^{N-d}) \leq \theta \). Therefore, \( \tilde{\phi}^{N-d} \in F_\theta \subseteq X_U \cap (\Omega_{i+d} \sim L_{i+d} \subseteq X_U \cap X^{N-d}) \). Finally, for \( i = N - d + 1 \), we note that we have just proven that \( h(\tilde{\phi}^{N-d}) \in U \) and \( \phi^{N-d+1} \in F_\theta = X_T \). Thus, \( \phi^{N-d+1} \in X_T \subseteq X^{N-d+1} \). To complete the proof, note that the shifted control input sequence is chosen as \( \tilde{u}_{d+k}, u_{d+1,k}, \ldots, u_{N-1,k}, h(\tilde{\phi}^{N-d}), \ldots, h(\tilde{\phi}^{N-1}) \). Thus, \( \tilde{\phi}^{N-d+2} \in X_T \subseteq X^{N-d+2} \). A similar argument holds for \( \tilde{\phi}^{N-d+3}, \tilde{\phi}^{N-d+4}, \ldots, \tilde{\phi}^{N} \). Thus, all the state constraints are satisfied.

Thus, we can always find a feasible control input sequence such that the problem (3) is feasible. Therefore, \( X_f(N) \) is a robust positive invariant set for up to \( N \) steps. Our final technical result is essential to prove the stability of the receding horizon system with packet dropouts and can be proven along the lines of (Quevedo and Nesic, 2011, Lemma 1).

**Lemma 9** If Assumption 1, 2, 4 are satisfied, then \( \forall x \in X_f(N), \forall \omega_{i-1}^{i} \in W^3, \forall i = 1, \ldots, N, \) there exists a \( K \) function \( \tilde{\gamma} \), such that \( V^*(f_i(x, \omega_{i-1})) - V^*(x) \leq -\alpha_1(\|x\|) + \gamma(\mu) \).

The following is the main result of the paper. Let \( \tilde{x}(k) \) be the actual state trajectory with packet dropouts. Note that \( \tilde{x}(k) \) is a function of \( d_{i-1}^{0}, w_{i-1}^{i-1}, \) and \( x(0) \).

**Theorem 10** Let Assumptions 1-4 be satisfied and \( \tilde{x}(k_0) \in X_f(N) \). Then there exist a \( K \) function \( \beta(\cdot) \) and a \( K \) function \( \gamma(\cdot) \), such that \( \| \tilde{x}(k) \| \leq \beta(\|\tilde{x}(k_0)\|, k - k_0) + \gamma(\mu), \forall k \geq k_0 \).

**PROOF.** To prove this result, first let us consider time instants \( k_0 \)'s at which the plant receives the control inputs from the controller. Because of Assumption 2.2, we obtain that \( \alpha_1(\|x\|) \leq V^*(x) \leq \alpha_2(\|x\|), \forall x \in X_f(N) \). Further, Proposition 8 and Lemma 9 ensure that \( \tilde{x}(k_i) \in X_f(N), \forall i \in Z_{\geq 0} \) and \( V^*(\tilde{x}(k_i)) - V^*(\tilde{x}(k_i)) \leq -\alpha_1(\|\tilde{x}(k_i)\|) + \gamma(\mu), \) where \( \gamma \) is a \( K \) function. Hence, by (Lazar and Heemels, 2009, Theorem 4), there exist \( K \) function \( \beta \) and \( K \) function \( \gamma \) such that \( \| \tilde{x}(k_i) \| \leq \beta(\|\tilde{x}(k_i)\|, k_i - k_0) + \gamma(\mu), \forall i \in Z_{\geq 0} \). For those time instants between \( k_i \) and \( k_i + 1 \), \( k_i \in Z_{\geq 0} \), we follow the argument in (Quevedo and Nesic, 2011, Theorem 1) to bound the state trajectory. The result then follows by noting that \( \| \tilde{x}(k) \| \leq \beta(\|\tilde{x}(k_0)\|, k - k_0) + \gamma(\mu), \forall k \geq k_0 \), where \( \beta(\cdot, \cdot) \) is a \( K \) function and \( \gamma(\cdot) \) is a \( K \) function.

4 Conclusions

This work has used the discrete-time piecewise continuous system to model hybrid systems. The remote controller design is based on receding horizon control. A communication channel between the controller and the plant can drop packets being transmitted. A buffer at the actuator stores the control input packets to compensate for such dropouts. We have analyzed input-to-state stability of such a system. The chief technical tool is the use of tightened state constraints and the design of the tightened margins to ensure the stability of the system.

References


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