Abstract—This paper considers the problem of disturbance propagation in a string of vehicles, where each vehicle has access to the position error with respect to its preceding vehicle. In addition, the followers in the string may receive coded information sent by the leader over finite capacity side channels. A lower bound on the integral of the sensitivity function of the position errors with respect to a stochastic disturbance acting on the lead vehicle is presented. This bound depends on the open-loop unstable modes of the system, as in the classical Bode integral formula for single-input single-output systems. However, in this case, the bound also depends on the capacities of the side communication channels. Simulation results illustrating the tightness of the proposed bound are presented.

I. INTRODUCTION

There has recently been extensive interest in formation control problems, where the objective is to arrange a large number of autonomous agents in a predefined geometric structure while ensuring that the agents as a group achieve a predefined task [1]–[4]. The simplest example is offered by the problem of controlling a one dimensional string of vehicles that has to move along a given trajectory while keeping constant spacing between any two consecutive vehicles. This problem arises, for instance, in the design of automated highway systems [5], [6].

It is known that the control performance when long strings of vehicles are being controlled in a distributed fashion depends on the amount of information available to each vehicle. In [2], it is shown that a “predecessor-following” control strategy, in which each vehicle has access to only its relative position with respect to the preceding vehicle, can cause error amplification along the string, eventually leading to string instability. This undesirable effect can be prevented under the “predecessor-leader” control strategy [7]–[9], wherein each vehicle is additionally given access to position information about the lead vehicle.

In this paper, we consider the formation control problem of a string for vehicles under the assumption that each vehicle has access to (i) its position error relative to the preceding vehicle, and (ii) to limited information about the platoon leader, which is communicated from the leader to the followers over capacity-limited side communication channels (see Fig. 1). The classical “predecessor-following” control strategy is recovered in the special case where the side channels have capacity zero, since no information can then be sent from the leader to the followers through such channels.

On the other hand, the “predecessor-leader” control strategy, in which the followers have perfect and instantaneous leader location information, can be recovered by assuming that the side channels have infinite capacity. In this paper, we study the sensitivity between the spacing errors along the string with respect to a stochastic disturbance acting on the lead vehicle and how this sensitivity changes as a function of the capacities of the side channels.

To deal with the stochasticity of the disturbance process and the presence of communication channels we follow the approach of Martins et al. [10]. Specifically, we use a notion of sensitivity which applies to stochastic processes and can be evaluated using tools from information theory. An additional advantage of this information-theoretic approach is that it allows us to directly relate the sensitivity to the Shannon capacities of the communication channels.

The main contribution of this paper is a lower bound for the integral of the sensitivity function of the position error at each vehicle with respect to the stochastic disturbance acting on the lead vehicle. This bound holds for a class of nonlinear controllers that includes, in particular, all linear time invariant (LTI) controllers. When there is only one follower in the string, our bound recovers the disturbance propagation result presented in [7]. Assuming instead that all side channels have zero capacity, our result recovers a previous bound derived in [11] under the “predecessor-following” control strategy.

We wish to point out some additional works that are related. Apart from the vast literature on string stability and platoon control, Shladover et al. [12] investigated the stability of a string formation under the assumption that all followers have immediate access to the outputs of both the leader and the immediate predecessor. The scenario where the state information is obtained after a fixed delay is considered in [13]. The degradation of tracking performance when the communication between the leader and the followers is lost is investigated in [14].

The rest of the paper is organized as follows. Section II provides a description of the problem. Section III states our main result and a discussion of its implications. Section IV focuses on the special case where the side communication channels are independent additive white Gaussian noise (AWGN) channels, in which case the sensitivity function is evaluated numerically and is compared to the derived lower bound. The proof of the main result of the paper is presented in Section V.

Notation: Throughout the paper, we denote random variables using boldface letters. For any $k \leq j$ we use the notation $x_k^j = (x(k), x(k + 1), \ldots, x(j))$ to denote finite
segment of a sequence \( x(1), x(2), \ldots \) and we omit the subscript \( k \) when this is equal to 1. Denote by \( \mathcal{U}(A) \) the set of unstable eigenvalues of an \( n \times n \) matrix \( A \).

II. Problem Formulation

Consider a string of \( n + 1 \) interconnected single-input single-output (SISO) vehicles as shown in Fig. 1, where \( P_i \)'s denote the plants and \( K_i \)'s represent the corresponding controllers \((0 \leq i \leq n)\). The state and the output of the \( i \)-th vehicle at time step \( k \) are denoted by \( x_i(k) \) and \( y_i(k) \), respectively. The string leader aims at following a reference command signal \( r_0(k) \) and its output \( y_0(k) \) is assumed to be perturbed by a stochastic disturbance \( d(k) \) in the feedback loop. Follower \( i \)'s goal is to regulate its output \( y_i(k) \) so as to satisfy \( y_{i-1}(k) - y_i(k) = \delta \) at every time step \( k \), where \( \delta \) is a prescribed constant corresponding to the desired spacing between successive vehicles. Thus, the spacing errors are defined as

\[
e_0(k) = r_0(k) + d(k) - y_0(k) \\
e_i(k) = y_{i-1}(k) - y_i(k) - \delta, \quad 1 \leq i \leq n.
\]

The dynamics of the \( i \)-th plant are given by:

\[
x_i(k) = A_i x_i(k-1) + B_i u_i(k-1) \\
y_i(k) = H_i x_i(k),
\]

where \( x_i(k) \in \mathbb{R}^{n_i}, y_i(k) \in \mathbb{R}, u_i(k-1) \in \mathbb{R} \) represents the control input, and \((A_i, H_i) \) is an observable pair.

We assume that vehicle \( i \) has access to the tracking error \( e_i \) and, possibly, to information sent from the leader over a side communication channel \( C_i \). Formally, \( C_i \) is defined as a channel input set \( \hat{X}_i \), a channel output set \( \hat{Y}_i \), and a family of transition probability mass functions \( p(\hat{y}_i | \hat{x}_i) \), for each \( \hat{x}_i \in \hat{X}_i \). We assume that the channel \( C_i \) is memoryless [15] and has Shannon capacity equal to \( C_i \), i.e.,

\[
C_i = \max_{p(\hat{x}_i)} I(\hat{X}_i; \hat{Y}_i),
\]

where \( I(\hat{X}_i; \hat{Y}_i) \) denotes the mutual information between \( \hat{X}_i \) and \( \hat{Y}_i \), which is defined as

\[
\sum_{\hat{x}_i, \hat{y}_i} p(\hat{y}_i | \hat{x}_i) p(\hat{x}_i) \log \frac{p(\hat{y}_i | \hat{x}_i) p(\hat{x}_i)}{\sum_{\hat{x}_i} p(\hat{y}_i | \hat{x}_i) p(\hat{x}_i)}.
\]

If the logarithm in the above expression is taken in base 2, then \( C_i \) is measured in units of bits per second and represents the maximum rate of communication across the channel \( C_i \).

At time \( k \), the controller of the \( i \)-th vehicle uses \( e_i^k, \hat{y}_i^k \) to generate a control signal of the form

\[
u_i(k) = u_{i,k}(\hat{y}_i^k, e_i^k).
\]

We assume that the control laws in (1) are such that the random processes describing the closed-loop dynamics have well defined continuous joint probability density functions and are asymptotically stationary processes. In addition, we make the following technical assumption.

**Assumption 1:** For every \((\hat{y}_i^k, e_i^{k-1}) \in \mathbb{R}^{2k+1}\), the nonlinear function \( u_{i,k}(\hat{y}_i^k, e_i^{k-1}, z) \) is a continuously differentiable bijective function of the variable \( z \in \mathbb{R} \).

As we will see later, this assumption is needed to relate the differential entropies of the random variables at the inputs and outputs of each plant.

For every \((\hat{y}_i^k, e_i^{k-1}) \in \mathbb{R}^{2k+1}\) we denote by \( u_{i,k}(\hat{y}_i^k, e_i^{k-1}, z) \) the partial derivative of (1) with respect to its last coordinate evaluated at \( e_i(k), i.e.,

\[
u_{i,k}^t(\hat{y}_i^k, e_i^{k-1}) := \frac{\partial}{\partial z} u_{i,k}(\hat{y}_i^k, e_i^{k-1}, z) \bigg|_{z=e_i(k)}.
\]

We denote by \( v_i \geq 1 \) the relative degree of the \( i \)-th plant, i.e., the smallest integer such that \( D_i := H_i A_i^{v_i-1} B_i \neq 0 \). By definition of relative degree, note that the control input \( u_i(k) \) does not affect the outputs \( y_i(k), \ldots, y_i(k+v_i-1) \), so \( v_i \) can be interpreted as the input-output delay introduced by the \( i \)-th plant.

Notice that \( |D_i u_{i,k}^t(e_i^k, \hat{y}_i^k)| \) characterizes the open-loop gain of the \( i \)-th subsystem at time step \( k \). As is well known, a large open-loop gain is desirable for good tracking performance. Therefore, we assume that the expected stationary value of \( |D_i u_{i,k}^t(e_i^k, \hat{y}_i^k)| \) is greater than 1. Define \( \Lambda_{i,k} := \log |D_i u_{i,k}^t(e_i^k, \hat{y}_i^k)| \) as the logarithm of the open-loop gain at time \( k \). Also define

\[
\Lambda_i := \log |D_i| + \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E} [\log |u_{i,k}^t(e_i^k, \hat{y}_i^k)|]
\]

as the corresponding stationary value, which will be used in the main result. We will assume for the most part that \( \Lambda_i > 0 \) for \( 0 \leq i \leq n \). The extension to the case of \( \Lambda_i < 0 \) is given in (5).

The notion of sensitivity function that is adopted in this paper is the same as the one used in [10], [16]:

![Fig. 1. A leader-follower platoon control system with side communication channels allowing the leader to send distinct coded information to each follower.](image-url)
Definition 1: The sensitivity function \( S_{x,y}(\omega) \) between the stationary stochastic processes \( x \) and \( y \) is defined as
\[
S_{x,y}(\omega) = \sqrt{\frac{\Phi_y(\omega)}{\Phi_x(\omega)}},
\]
where \( \Phi_x(\omega) \) and \( \Phi_y(\omega) \) are the power spectral densities of \( x \) and \( y \), respectively.

We refer the reader to [10] for a discussion on the relationship between \( S_{x,y}(\omega) \) and the classical input-output sensitivity function and on the main properties of \( S_{x,y}(\omega) \).

The objective of this paper is to investigate the sensitivity functions. Specifically, we suppose that \( x \) and \( y \) are random variables involved are jointly Gaussian, because in this case it is possible to provide closed-form expressions for the sensitivity functions. Specifically, we suppose that \( d = G(\omega)\nu \), where \( G(\omega) \) is a known stable all-pole linear filter and \( \nu \) is an independent identically distributed (i.i.d.) Gaussian process with unit variance. Moreover, we suppose that \( C_i \) is an additive white Gaussian noise (AWGN) channel, for which the channel input \( \hat{x}_i \) and the channel output \( \hat{y}_i \) are real numbers related as follows
\[
\hat{y}_i = \hat{x}_i + \omega_i.
\]

III. MAIN RESULT

The main result of the paper is the following.

Theorem 1: Let Assumptions 1-3 hold. Then, for every \( i = 0, \ldots, n \),
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_i}(\omega) d\omega \geq \sum_{j=0}^{i-1} \max(\Lambda_j - C_j, 0) + \sum_{\lambda \in U(A_i)} \log |\lambda| - C_i.
\]

The proof of Theorem 1 is presented in Section V. A few remarks about the result are in order.

1) Similar to the Bode integral formula for SISO plants, the right hand side of (4) depends on the \( i \)-th plant’s unstable modes and is independent of the choice made for the \( i \)-th controller. Therefore, (4) characterizes a fundamental limitation for all control laws satisfying Assumption 1.

2) In the special case where \( C_j = 0 \) for all \( 0 \leq j \leq i - 1 \), Theorem 1 recovers the result in [11] with the predecessor-following control strategy.

3) In the special case where \( \Phi_{e_2}(\omega) \) of (1) and \( \Phi_{e_1}(\omega) \) of (2) simplify to \( F_i(\omega) \) and \( F_i(\omega) \) respectively.

4) Suppose \( \Lambda_m < 0 \) where \( 0 \leq m \leq i - 1 \), meaning that the open-loop gain for the \( m \)-th subsystem is smaller than 1. Then the inequality (4) is replaced by
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_i}(\omega) d\omega \geq \sum_{j=0}^{m} \max(\Lambda_j - C_j, 0) + \sum_{j=m+1}^{i-1} \max(\Lambda_j - C_j, 0) + \Lambda_m + \sum_{\lambda \in U(A_i)} \log |\lambda| - C_i.
\]

The proof is similar to the \( \Lambda_m > 0 \) case and omitted because of space constraints.

IV. EXAMPLE: SIDE INFORMATION OVER GAUSSIAN CHANNELS

In this section, we focus on the special case where all the random variables involved are jointly Gaussian, because in this case it is possible to provide closed-form expressions for the sensitivity functions. Specifically, we suppose that \( d = G(\omega)\nu \), where \( G(\omega) \) is a known stable all-pole linear filter and \( \nu \) is an independent identically distributed (i.i.d.) Gaussian process with unit variance. Moreover, we suppose that \( C_i \) is an additive white Gaussian noise (AWGN) channel, for which the channel input \( \hat{x}_i \) and the channel output \( \hat{y}_i \) are real numbers related as follows
\[
\hat{y}_i = \hat{x}_i + \omega_i.
\]

where \( \omega_i \) is a zero-mean Gaussian random variable independent of \( \hat{x}_i \), with variance \( \sigma_\omega^2 \). In addition, the channel input is subject to the power constraint \( \mathbb{E}(|\hat{x}_i|^2) \leq \sigma_p^2 \). It is known [15] that the capacity of the channel in (6) under such a power constraint is equal to \( C_i = \frac{1}{2} \log(1 + \sigma_p^2 / \sigma_\omega^2) \).

Since the channel capacity only depends on the ratio of \( \sigma_p^2 / \sigma_\omega^2 \), there is no loss of generality by assuming that all \( \omega_i \)'s share a common variance \( \sigma_\omega^2 \).

Next, we suppose that the encoding function \( E_i \) that maps \( y_0^k \) into the channel input \( \hat{x}_i(k) \) is given by
\[
E_i(y_0) = z^{-v_0} G^{-1}(\omega) T_0^{-1}(\omega) \sigma_p \cdot y_0.
\]

Here, \( T_i(\omega) \) denotes the complementary sensitivity function of the \( i \)-th system. Notice the term \( z^{-v_0} \) ensures that the encoding mapping is causal. The control mapping is chosen as
\[
\Phi_{e_i}(\omega) = \frac{\sigma_p^2}{\sigma_p^2 + \sigma_\omega^2} |L_2 T_1 T_0(\omega)|^2 \Phi_d(\omega),
\]

where \( F_i(\omega) \) is a stable linear and time-invariant stabilizing controller for the \( i \)-th subsystem.

Under these assumptions, the power spectral densities of the spacing errors and the disturbance can be derived explicitly. First, let us consider the second follower. We have
where $L_i(\omega)$ denotes the sensitivity function of the $i$-th system, $i = 0, 1, \ldots, n$. It follows that the sensitivity function from the disturbance $d$ to $e_2$ can be written as
\[
S_{d,e_2}(\omega) = \sqrt{\Phi_{e_2}(\omega)/\Phi_d(\omega)} = 2^{-C_2}|L_2T_1T_0(\omega)|. \quad (9)
\]
Thus
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_2}(\omega)d\omega = \sum_{\lambda \in \mathcal{U}(A_2)} \log |\lambda| + \frac{1}{2} \sum_{j=0}^{1} \Lambda_j - C_2.
\]

It is worth pointing out that the lower bound in (4) is, thus, achieved with equality at the second follower. Observe from (9) that the sensitivity $S_{d,e_2}(\omega)$ decreases exponentially with the capacity $C_2$ at all frequencies. Accordingly, the integral of the log sensitivity function reduces linearly as $C_2$ increases and thus tends to $-\infty$ as $C_2 \to \infty$. In this case, follower 2 can perfectly predict the disturbance $d$.

Next, let us consider the third follower. By following a similar argument as above, it is possible to show that
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_3}(\omega)d\omega = \sum_{\lambda \in \mathcal{U}(A_3)} \log |\lambda| + \frac{1}{4} \sum_{j=0}^{1} \Lambda_j + \frac{1}{4\pi} \int_{-\pi}^{\pi} \log |\lambda|d\omega,
\]
where
\[
|\Omega(\omega)| = 2^{-2(C_2+C_3)}(T_2^2(\omega) - 1) + 2^{-2C_3}
+ 2^{-2C_2}(1 - 2^{-2C_2})(1 - 2^{-2C_3})L_2^2(\omega).
\]

The analysis becomes cumbersome as we go beyond the third follower. For this reason we resort to numerical methods to compare the integral of the log sensitivity function with the theoretical lower bound in Theorem 1. A numerical example for $n = 3$ where linear controllers and AWGN channels as above are used is considered to illustrate the theoretical results in Section III. For the sake of simplicity, it is assumed that all the members in the platoon have identical plants and controllers. The open-loop plant and the feedback controller are given as $P(z) = \frac{1}{z^2}$ and $K(z) = \frac{2(z+0.52)}{z+0.2}$, respectively and the close-loop poles are placed at $-0.1 \pm 0.8j$ as a result. Moreover, it can be obtained that $\sum_{\lambda \in \mathcal{U}(A_i)} \log |\lambda| = 1$ and $\Lambda_1 = 1$ for $0 \leq i \leq n$.

The lower bound in (4) is compared with the disturbance propagation performance in Fig. 2 (a) and (b) for to different values of $C_2$ and $C_3$. There are a couple of points that need to be emphasized:

1. Fig. 2 (a) and (b) illustrate that it is not possible to arbitrarily reduce the integral of the log sensitivity function at follower 3 by increasing the capacity $C_3$ while keeping $C_2$ constant, nor by increasing the capacity $C_2$ while keeping $C_3$ constant.

2. The bound (4) is tight for follower 3 in the special cases where $C_2 = 0$ or $C_2 \to \infty$. This can be seen in Fig. 2 (a) and (b), where the lower bound in (4) is asymptotically tight in the limit as $C_2 \to \infty$.

V. PROOF OF THEOREM 1

In order to prove Theorem 1, we first establish four auxiliary lemmas based on information-theoretic arguments. We refer the reader to [15, Chapter 2] for an introduction to basic information-theoretic quantities and their properties. We concentrate on the case where $2 \leq i \leq n$, since the result for $i = 0$ and $i = 1$ is proved in [11].
Lemma 1: Consider the set-up described in Section II. Let Assumptions 2 and 3 hold. Then, for $2 \leq i \leq n$,
\[
\lim_{k \to \infty} \frac{1}{k} I(e_i^{k-1}; x_i(0)|\hat{y}_i^{k-1}) \geq \sum_{\lambda \in \mathcal{L}(A_i)} \log |\lambda|.
\]  
(11)

Proof: Since $d$, $x_0(0)$ and $x_i(0)$ ($2 \leq i \leq n$) are mutually independent, it is easy to see that $\hat{y}_i$, which is a function of $d$ and $x_0(0)$, is also independent of $x_i(0)$, i.e., $I(\hat{y}_i; x_i(0)) = 0$. Thus, by the chain rule of mutual information,
\[
I(e_i^{k-1}; x_i(0)|\hat{y}_i^{k-1}) = I((e_i^{k-1}, \hat{y}_i^{k-1}); x_i(0)) \geq I(u_i^{k-1}; x_i(0))
\]
where (a) follows from the data processing inequality. Therefore,
\[
\lim_{k \to \infty} \frac{1}{k} I(e_i^{k-1}; x_i(0)|\hat{y}_i^{k-1}) \geq \lim_{k \to \infty} \frac{1}{k} I(u_i^{k-1}; x_i(0)) = \sum_{j=1}^{b} \max \{0, \log |\lambda_j(A_i)|\}
\]
where (b) follows from [10, Lemma 4.1].

Loosely speaking, Lemma 1 states that in order to keep the $i$th closed loop system stable, the rate at which the initial condition $x_i(0)$ can be learned from the spacing error signal $e_i^{k}$ must be greater than the rate of expansion of the open-loop system. This result is reminiscent of the data rate theorem [20] for stabilization of unstable plants over a communication constrained feedback channel, which states that the information rate to be supported by the feedback channel to keep the system stable must be large enough compared to the unstable modes of the system, so that it can compensate for the expansion of the state during the communication process.

Lemma 2: For $2 \leq i \leq n$,
\[
h(e_i(k)|e_i^{k-1}, x_i(0), \hat{y}_i^{k-v_i}) = h(e_i(k)|e_i^{k-1}, x_i(0), \hat{y}_i^{k})
\]
for every $N \geq k - v_i$.

Proof: We have
\[
h(e_i(k)|e_i^{k-1}, x_i(0), \hat{y}_i^{k}) = h(e_i(k)|\hat{y}_i, x_i(0), \hat{y}_i^{k-v_i}) = I(e_i(k); \hat{y}_i^{k-v_i}) + h(\hat{y}_i^{k-v_i})
\]
and
\[
h(e_i(k)|e_i^{k-1}, x_i(0), \hat{y}_i^{k-v_i}) = h(e_i(k)|\hat{y}_i^{k-v_i}) + h(\hat{y}_i^{k-v_i})
\]
Notice that, since there is no feedback from the followers to the leader, $(e_i^{k}, x_i(0)) \rightarrow \hat{y}_i^{k-v_i} \rightarrow \hat{y}_i^{N}$, $i=1, \ldots, n$ form a Markov chain. As a result,
\[
h(\hat{y}_i^{N}) = h(\hat{y}_i^{k-v_i})
\]
Therefore, Lemma 2 follows by substituting the above equation into (12).

Lemma 2 states that information about the leader’s position at future times, i.e., $\hat{y}_i^{k-v_i}$, does not affect the uncertainty about the spacing error $e_i(k)$ at time $k$.

Lemma 3: Let Assumptions 2 and 3 hold. Then, for $2 \leq i \leq n$,
\[
\bar{h}(d_i) \geq \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} h(y_{i-1}(k)|y_{i-1}^{k-v_i}, y_{i-2}^{k-v_i-1}, \hat{y}_i^{k-v_i}) + \bar{h}(d_i)
\]
\[
\geq \sum_{j=1}^{b} \max \{0, \log |\lambda_j(A_i)|\}
\]
where (a) follows from the data processing inequality. Therefore, Lemma 2 follows by substituting the above equation into (12).

Proof: For every $N > k - v_i$,
\[
h(e_i(k)|e_i^{k-1}) = I(e_i(k); \hat{y}_i^{k-v_i})
\]
\[
\geq h(e_i(k)|e_i^{k-1}, x_i(0), \hat{y}_i^{k-v_i})
\]
\[
\geq h(e_i(k)|e_i^{k-1}, x_i(0), y_i^{k-v_i})
\]
\[
\geq h(e_i(k)|e_i^{k-1}, y_i^{k-v_i})
\]
\[
\geq h(e_i(k)|y_i^{k-v_i})
\]
\[
\geq h(y_{i-1}(k)|y_{i-1}^{k-v_i-1}, y_i^{k-v_i})
\]
\[
\geq h(y_{i-1}(k)|y_{i-1}^{k-v_i-1}, y_i^{k-v_i})
\]
\[
= h(y_{i-1}(k)|y_{i-1}^{k-v_i-1}, y_i^{k-v_i})
\]
where (a) follows from Lemma 2, (b) holds because $y_i^k$ is a function of $e_i^{k}$, $\hat{y}_i^{k-v_i}$ and $x_i(0)$, (c) follows from the observability of the $i$-th subsystem, (d) follows from the fact that $e_i(k) = y_{i-1}(k) - y_i(k) - \delta$, the equality in (e) holds when $y_{i-1}^{k-v_i} \rightarrow (y_{i-1}^{k-v_i}, y_i^{k-v_i}) \rightarrow y_{i-1}(k)$ form a Markov chain. (f) is true because $y_{i-1}(k) \rightarrow (y_{i-1}^{k-v_i}, y_i^{k-v_i}) \rightarrow y_i^{k-v_i}$ form a Markov chain. Then, (13) is obtained by summing both sides of (14) over $k$, using the chain rule, dividing by $N$, and taking the limit as $N \to \infty$.

Next, we state a result that makes explicit the effect of the side communication channels. We omit the proof because of space limitation.

Lemma 4: Let Assumptions 1-3 hold. Then, for $2 \leq i \leq n$,
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} h(y_{i}(k)|y_{i}^{k-v_i-1}, y_{i-1}^{k-v_i-1}, \hat{y}_i^{k-v_i})
\]
\[
\geq \sum_{j=1}^{b} \max(A_j - C_j, 0) + \bar{h}(d) - C_{i+1}.
\]

Equipped with these auxiliary results, we can now prove Theorem 1.
Proof of Theorem 1: For $2 \leq i \leq n$

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_i}(\omega) \, d\omega
$$

$$
= \frac{1}{4\pi} \int_{-\pi}^{\pi} \log \left( \frac{2\pi e^{\bar{\Phi}_{d}(\omega)}}{\bar{\Phi}_{d}(\omega)} \right) \, d\omega
$$

$\geq h(e_i) - h(d)$

$\geq \sum_{j=0}^{i-1} \max(A_j - C_j, 0) + \lim_{N \to \infty} \frac{1}{N} I(e_i^{N-1}; x_i(0)|\hat{y}_{i}^{N-1})$

$$
+ I(e_i; y_i) - C_i
$$

$\geq \sum_{j=0}^{i-1} \max(A_j - C_j, 0) + \sum_{\lambda \in \mathcal{U}(A_i)} \log |\lambda| - C_i,$

where (a) follows from the maximum entropy theorem and the equality holds when $e_i$ is Gaussian, (b) follows from Lemma 3 and Lemma 4, and (c) follows from Lemma 1 and the positivity of mutual information. \(\square\)

VI. Conclusion

In this paper we investigated a leader-follower platoon control problem with non-cyclic information pattern, where the leader can communicate to the followers via capacity-limited side channels. A lower bound on the integral of the log sensitivity function was derived using information-theoretic arguments, and the effect of the side communication channels was discussed. The result in this paper applies to a class of nonlinear controllers. A numerical example was provided to show the usefulness of the theoretical results.

This work dealt with control of string systems with unidirectional information flow, from the leader to the followers. Future work includes the extension to the case where the information flow is cyclic, so a follower can communicate backwards with its predecessors. The fundamental question that is leitmotif of our future research is how to generalize Bode’s integral formula to general distributed systems. This generalization is currently available only in few special cases.

REFERENCES


